

Uniqueness of Kottler spacetime and Besse conjecture

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Abstract

We establish a black hole uniqueness theorem for Schwarzschild-de Sitter spacetime, also called Kottler spacetime, which satisfies Einstein's field equations of general relativity with positive cosmological constant. Our result concerns the class of static vacuum spacetimes with compact spacelike slices and regular maximal level set of the lapse function. We provide a characterization of the interior domain of communication of the Kottler spacetimes, which surrounds an inner horizon and is surrounded by a cosmological horizon. The proposed proof combines arguments from the theory of partial differential equations and differential geometry, and is centered on a detailed study of a possibly singular foliation. We also apply our technique in the Riemannian setting, and establish the validity of the so-called Besse conjecture.

1. Introduction

Static vacuum spacetimes satisfying the Einstein field equations play a central role in general relativity, since such spacetimes are expected to represent a final state of the evolution of matter under self-gravitating forces. Several classical results show that, under certain physical conditions, a very limited number of such spacetimes exists. We are interested here in the class of spatially compact spacetimes with positive cosmological constant, which is not covered by the mathematical techniques available in the literature and, therefore, we establish here a new black hole uniqueness theorem. Our proof overcomes several conceptual and technical difficulties, as explained below.

By definition, a *static spacetime with maximal compact spacelike slices* (of class $W^{2,2}(\overline{\mathcal{M}})$) is a time-oriented, $(3 + 1)$ -dimensional Lorentzian manifold \mathbf{N} with global topology $\mathbf{N} \simeq \mathbb{R} \times \mathcal{M}$ and Lorentzian metric $\mathbf{g} = -f^2 dt^2 + g$, where t is a coordinate on \mathbb{R} increasing toward the future, \mathcal{M} is a connected,

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orientable, smooth topological 3-manifold with smooth boundary $\partial\mathcal{M}$ such that $\overline{\mathcal{M}} := \mathcal{M} \cup \partial\mathcal{M}$ is compact and is endowed with a t -independent Riemannian metric g of class $W^{2,2}(\mathcal{M})$, and $f : \mathcal{M} \rightarrow (0, +\infty)$ belongs to the Sobolev space $W^{2,2}(\mathcal{M})$ and vanishes at the boundary.

The assumed regularity means that, in an atlas of local coordinates, the metric coefficients admit derivatives up to second-order that are squared-integrable. In this context, f is referred to as the *lapse function*, and the vector field $\mathbf{T} := \partial/\partial t$ is a future-oriented, timelike Killing field:

$$\mathcal{L}_{\mathbf{T}}g = 0, \quad g(\mathbf{T}, \mathbf{T}) < 0.$$

By definition, the hypersurfaces $t = \text{const.}$ are orthogonal to \mathbf{T} , and the spacetime \mathbf{N} is foliated by compact spacelike slices with boundary. The lapse function is positive in \mathcal{M} and vanishes on $\partial\mathcal{M}$, so that the zero-level set of f

$$\mathcal{H} := \{f = 0\},$$

referred to as the *horizon*, coincides with the boundary of the slices $\mathcal{H} = \partial\mathcal{M}$ (which need not be connected).

In addition, we impose that \mathbf{N} satisfies Einstein's vacuum equations with positive cosmological constant $\Lambda > 0$, that is, $\mathbf{G}_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, where $\mathbf{G}_{\mu\nu} := \mathbf{R}_{\mu\nu} - (\mathbf{R}/2)g_{\mu\nu}$ denotes Einstein's curvature tensor (in dimensions $3+1$), $\mathbf{R}_{\mu\nu}$ the Ricci curvature, and \mathbf{R} the scalar curvature, respectively. In other words, we impose

$$\mathbf{R}_{\mu\nu} = \Lambda g_{\mu\nu}.$$

Such a spacetime was discovered by Kottler [6], and its most relevant part for us is the “interior domain of communication”, defined as follows. Given $m, \Lambda > 0$ satisfying $(3m)^2\Lambda \in (0, 1)$, the *interior domain of the Kottler spacetime*, denoted by $\mathbf{N}_{\mathcal{K}, m, \Lambda}$ with metric $g_{\mathcal{K}, m, \Lambda}$, is the static spacetime with maximal compact spacelike slices, whose lapse function $f_{\mathcal{K}, m, \Lambda}$ and Riemannian metric $g_{\mathcal{K}, m, \Lambda}$ on the compact spacelike slices

$$\mathcal{M}_{\mathcal{K}, m, \Lambda} \simeq (r_{\mathcal{K}}^-, r_{\mathcal{K}}^+) \times S^2$$

are defined by

$$(f_{\mathcal{K}, m, \Lambda}(r))^2 := 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2, \quad g_{\mathcal{K}, m, \Lambda} := \frac{dr^2}{(f_{\mathcal{K}, m, \Lambda}(r))^2} + r^2 g_{S^2}, \quad r \in [r_{\mathcal{K}}^-, r_{\mathcal{K}}^+],$$

where g_{S^2} denotes the canonical metric on the unit sphere S^2 , m is interpreted as the mass of the spacetime, and $r_{\mathcal{K}}^\pm = r_{\mathcal{K}^\pm, m, \Lambda}$ are the two positive roots of the cubic polynomial $r \mapsto r(f_{\mathcal{K}, m, \Lambda}(r))^2$.

These manifolds are also called *Schwarzschild-de Sitter spacetimes* and provide us with a two-parameter family of static spacetimes with compact spacelike slices, which are locally (but not globally) conformally flat. Note that the horizon of a Kottler spacetime, denoted here by $\mathcal{H}_{\mathcal{K}, m, \Lambda}$, consists of the two connected components

$$\mathcal{H}_{\mathcal{K}, m, \Lambda}^\pm := \{r = r_{\mathcal{K}^\pm, m, \Lambda}\}.$$

We point out that the spacetimes $\mathbf{N}_{\mathcal{K}, m, \Lambda}$ may be extended beyond their horizon: one component of $\mathcal{H}_{\mathcal{K}, m, \Lambda}$ is an “inner horizon” connecting to an interior black hole region while the other component is a cosmological horizon connecting to a non-compact exterior domain of communication (asymptotic to de Sitter). The interior domain is, both mathematically and physically, the region of interest and for instance, as $\Lambda \rightarrow 0$, converges to the outer communication domain of the Schwarzschild spacetime dealt with in the classical black hole theorems.

Finally, one more family of spacetimes are relevant in the present work, that is, the *de Sitter spacetimes*, parametrized by their cosmological constant $\Lambda > 0$. We denote by $\mathbf{N}_{dS, \Lambda}$ one domain of communication of the de Sitter spacetime, whose spacelike slices have the topology of a half-sphere S^3 and whose horizon $\mathcal{H}_{dS, \Lambda}$ admits a single component diffeomorphic to the 2-sphere S^2 .

2. Main results

We are now in a position to state our rigidity results, under the regularity condition that the level set achieving the maximum of the lapse function is a regular surface.

Theorem 1 (Uniqueness theorem for Kottler spacetime) *The interior domain of the Kottler spacetimes $\mathbf{N}_{\mathcal{K},m,\Lambda}$ parameterized by their mass $m > 0$ and cosmological constant $\Lambda > 0$ together with the domain of communication $\mathbf{N}_{dS,\Lambda}$ of the de Sitter spacetimes are, up to global isometries, the unique static spacetimes with maximal compact spacelike slices and regular maximal level set, satisfying Einstein's field equations with positive cosmological constant.*

We emphasize that no restriction is assumed a priori on the topology of the spacelike slices, and this topology is finally identified as part of the conclusion of the theorem, which also provides us with the metric. Hence, the above theorem is of interest in both general relativity and topology. A large literature is available on black hole uniqueness theorems, and we will not try to review it here but will only quote works that are most related to the present discussion.

Classical works deal with the case $\Lambda = 0$, and goes back to Israel [4], Hawking [3], and many others. For more recent works, see Lindblom [10] and Beig and Simon [1]. The class of (vacuum) spacetimes with negative cosmological constant $\Lambda < 0$ was tackled only recently. (See [8] for references.)

In contrast with the above results and despite active research on the subject in the past twenty years, the class of spacetimes with positive cosmological constant is not amenable to the mathematical techniques developed in the existing literature. Our purpose in the present paper is to introduce a new approach which overcomes these (technical and conceptual) difficulties and to establish a uniqueness theorem for the case $\Lambda > 0$. As we will show, we have to combine arguments from partial differential equations and differential geometry, and, most importantly, to work within a class of possibly singular foliations.

Our method of proof also applies in the Riemannian setting and allows us to establish the validity of Besse conjecture [2]. (See also the earlier works [5,7] for special cases.)

Theorem 2 (Besse conjecture in Riemannian geometry) *All compact three-manifolds (M, g) , on which there exists a non-trivial solution f to the dual linearized curvature equation $L^*(f) = 0$ with regular maximal level set, are given by the following list (up to isometries):*

- *The sphere S^3 endowed with the canonical metric. In this case, one has $f = \cos(d(\cdot, x_0))$ where d is the Riemannian distance to a point x_0 , and the kernel of L^* has dimension $\dim \text{Ker}(L^*) = 4$.*
- *A finite quotient of the product $S^1 \times S^2$ endowed with the canonical product metric. In this case one has $\dim \text{Ker}(L^*) = 2$.*
- *A finite quotient of the twisted product $S^1 \times S^2$ endowed with the metric $g = dx^2 + h^2(x) g_{S^2}$. These twisted products depend upon two real parameters and an integer parameter, and $\text{Ker}(L^*) = h' \mathbb{R}$.*

3. Elements of proof

Let us indicate several key elements of our proof of Theorem 1. We consider a static spacetime \mathbf{N} with maximal compact spacelike slices \mathcal{M} (and $W^{2,2}$ regularity) satisfying Einstein's field equations with positive cosmological constant $\Lambda > 0$. Using the $(3+1)$ -splitting, the Einstein equations on the 4-dimensional spacetime are equivalent to a problem posed on the 3-manifold \mathcal{M} with boundary, i.e., to the partial differential equations (for the lapse function f and metric g)

$$\nabla df - (\Delta f) g - f Rc = 0,$$

with the additional constraint that the scalar curvature R of (\mathcal{M}, g) coincides with) the cosmological constant and, therefore, is a constant; specifically, one has $R = 2\Lambda > 0$. In the Einstein equations, the field of 1-forms df is the differential of f , while ∇ denotes the covariant derivative in (\mathcal{M}, g) , ∇df the Hessian of f , Δ the Laplacian operator (normalized to have negative eigenvalues), and Rc the 3-dimensional Ricci curvature, respectively. By taking the trace of the Einstein equations, we deduce that

$$\Delta f = -\frac{R}{2}f.$$

In other words, f is an eigenfunction of the Laplace operator defined on the (unknown) Riemannian manifold (\mathcal{M}, g) . Our objective is to determine *all triplets of solutions* (\mathcal{M}, g, f) satisfying the Einstein equations and, in particular, to determine the topology of \mathcal{M} .

From the lapse function associated with the natural $(3+1)$ -foliation of the spacetimes under consideration, we define a (possibly) degenerate $(2+1)$ -foliation and investigate the topology and geometry of its leaves. It is convenient to introduce certain *normalized* geometric invariants of this foliation, which make sense globally on the manifold \mathcal{M} , even at points where the gradient ∇f vanishes (and the foliation possibly becomes degenerate). We also introduce the *Hawking mass density*, defined from the Gauss curvature and mean-curvature of the 2-slices, which again makes sense globally on the manifold, even at critical points. The Hawking mass density, used here, appears classically as an integrant in Hawking's original definition. Using the notion of Hawking mass density, we establish a pointwise version of Penrose inequality on the horizon, which allows us to identify a topological 2-sphere within the connected components of the horizon. An "optimal" Kottler model with well-chosen ADM mass is introduced, which covers the region limited by certain level sets of the lapse function. Finally, several maximum principle arguments are developed for Einstein's field equations of static spacetimes, which apply to the possibly degenerate $(2+1)$ -foliation under consideration. For further details we refer to [8,9].

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